Homework 10

Consider the transient, one-dimensional heat conduction problem considered in the class with the boundaries insulated and an initial temperature distribution given by $f(x)$.

The solution to this problem was shown to be

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\alpha\beta_n^2 t} cos\beta_n x, \quad \beta_n = \frac{n\pi}{l}.$$

Therefore, initial condition satisfies $f(x)$ satisfies:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\beta_n x$$

Note that when $f(x)$ is expressed as above, it is an even function on the interval $[-l, l]$.

The constants $a_n, n = 0, 1, 2, 3, \ldots$ were determined to be

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{1}{l}\int_0^l f(x)dx$$

and

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\beta_n x dx, = \frac{2}{l}\int_0^l f(x)\cos\beta_n x dx, \text{ for } n = 1, 2, 3, \cdots.$$

1. Suppose that $f(x)$ has the following form:

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{l}{4}, \\ 1600(\frac{l}{4} - x)(x - \frac{3l}{4}), & \frac{l}{4} \le x \le \frac{3l}{4}, \\ 0, & \frac{3l}{4} \le x \le l. \end{cases}$$

   Plot the term m-term Fourier approximation of $f(x)$ using $m = 1, 2, 3, 5, 10$ and 50. What is a reasonable number for the maximum number of terms for representing $f(x)$.

2. For the function $f(x)$ of Problem 1, conduct an energy balance and show that a_0 represents the steady-state solution.

3. For the function $f(x)$ of Problem 1, plot the temperature distribution in the bar using a truncated Fourier series with m terms for $t = 0, 10, 20, 100$ and 500 seconds. Use a value of m that gives reasonably accurate results.

4. Consider the steady-state heat conduction in a rectangular plate of dimensions $b \times h$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the boundary conditions:

$$\begin{aligned} u(0,y) &= 0 \\ u(b,y) &= 0 \\ u(x,0) &= 0 \\ u(x,h) &= Ax\sin x. \end{aligned}$$

Using the method of separation of variables, obtain the temperature distribution in the plate. Assuming that the plate is made of steel, $b = 0.2$ m and $h = 0.4$ m, provide a contour plot of the temperature distribution in the plate. A three-dimensional plot is also acceptable. For plotting purposes, plot $u(x,y)/A$.