PhD Qualifying Exam (Math)

Spring, 2023

Name: ____

Question:	1	2	3	4	5	Total
Points:	25	25	25	25	25	100
Score:						

Instructions:

- Solve only four of the five problems
- Show all your work; no points are given if you solely provide the final solution.
- Please return the sheets along with your answer sheets.
- Do not make any assumptions that change the nature of the problem.

PhD Qualifying Exam (Math) Spring, 2023

Problem 1 – Partial Differential Equation (25 Points)

Consider the following partial differential equation cantilever-beam problem:

$$u_t - u_{xx} + u = 0$$

with

$$u(0, t) = 0,$$

$$u(1, t) = 0,$$

and

$$u(x, 0) = \sin(\pi x)$$

where *0 ≤ x ≤ 1, and 0 ≤ t < ∞*.

(a) Derive the separated coupled ordinary differential equations using the principle of separation of variables $u(x, t) = X(x)T(t)$. (7 points)

(b) Find the general solution to the ordinary differential equations for $X(x)$ and $T(t)$. (3 points)

(c) Apply the boundary conditions $u(0, t) = 0$ and $u(1, t) = 0$ to estimate the value of the unknown constants of the general solution that was estimated in (b). (8 points)

(d) Calculate the remaining coefficients for the initial value problem $u(x, 0) = \sin(\pi x)$ (3 points)

(e) Draw (plot) the solution for $u(x, 1)$ in the interval $x = [0, 6\pi]$ (2 points)

(f) Draw (plot) the solution for $u\left(\frac{3}{2}, t\right)$ in the interval $t = [0, 2]$ (2 points)

PhD Qualifying Exam (Math) Spring, 2023

Problem 2 – Power Series Solutions (25 Points)

Consider the differential equation

$$y'' - xy = 0$$

(a) Determine whether the differential equation ha s singular points. Explain (2 points)

(b) Determine whether x = 0 is an ordinary point of the differential equation. Explain. (2 points)

(c) Find a recurrence formula for the power series solution around x = 0 for the differential equation (5 points)

(d) Express the coefficients a_2 to a_8 as a function of a_0 or a_1. (7 points)

(e) Use the recurrence formula of the power series solution to find analytical expressions for (i) possible solutions of the differential equation, and (ii) the superposition of all possible solutions. (9 points)

Problem 3 – Method of underdetermined coefficients (25 Points)

Consider the following ordinary differential equation

$$y'' + 2y' + 0.75y = 2\cos(x)$$

(a) Provide the homogenous solution (max 6.5 points).

(b) Provide the particular solution using the method of underdetermined coefficients. Calculate all unknown coefficients of this particular solution (max 6.5 points).

Now consider the following ordinary differential equation

$$y''' + 3y'' + 3y' + y = e^x - x - 1$$

with the characteristic polynomial $(\lambda + 1)^3 = 0$.

(c) Provide the homogenous solution (max 4 points).

(d) Provide the particular solution using the method of underdetermined coefficients. Calculate all unknown coefficients of this particular solution (max 8 points).

Problem 4 – Exact Equations & Integrating Factors (25 Points)

Consider the following first order ordinary differential equation for y(x)

$$5y^2x^2\frac{dy}{dx} - \frac{5}{3}y^3x - \frac{1}{x^2} = 0$$

(a) Show that the above ODE is not exact by checking the exactness condition (max 6 points).

(b) Transform the ODE into an exact ODE using an integrating factor (max 7 points).

(c) Solve for the implicit general solution of the exact ODE from part b) (max 8 points).

(d) Solve the explicit particular solution with the initial condition y(1)=-1 (max 4 points).

PhD Qualifying Exam (Math)

Spring, 2023

Problem 5 – Reduction of Order (25 Points)

Consider the following first order ordinary differential equation (ODE) for $y(x)$ with $x > 0$:

$$2xy'' - 3y' + \frac{2y}{x} = 0$$

Suppose that one solution is known to be $y_1(x) = x^2$

- **(a)** Verify that $y_1(x)$ is indeed a solution to the ODE (max 4 points).
- **(b)** Use the reduction of order method to find a second solution $y_2(x)$ (max 15 points).
- **(c)** Show that $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions (max 6 points).

TABLE OF INTEGRALS

Substitution Rule	Integration by Parts
$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad (u = g(x))$	$\int u\,dv = uv - \int v\,du$
$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$	$\int_a^b uv'\,dx = uv\Big\rvert_a^b - \int_a^b vu'\,dx$

Basic Integrals

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C;\ n \neq -1$

2. $\int \frac{dx}{x} = \ln|x| + C$

3. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

4. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

5. $\int \tan x\,dx = \ln|\sec x| + C$

6. $\int \cot x\,dx = \ln|\sin x| + C$

7. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

8. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

9. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

10. $\int b^{ax}\,dx = \frac{1}{a\ln b}b^{ax} + C;\ b > 0, b \neq 1$

11. $\int \ln x\,dx = x\ln x - x + C$

12. $\int \log_b x\,dx = \frac{1}{\ln b}(x\ln x - x) + C$

13. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

14. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

15. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

16. $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1 - x^2} + C$

17. $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1 - x^2} + C$

18. $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C$

19. $\int \sec^{-1}x\,dx = x\sec^{-1}x - \ln(x + \sqrt{x^2 - 1}) + C$

20. $\int \sinh x\,dx = \cosh x + C$

21. $\int \cosh x\,dx = \sinh x + C$

22. $\int \operatorname{sech}^2 x\,dx = \tanh x + C$

23. $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

24. $\int \operatorname{sech} x\tanh x\,dx = -\operatorname{sech} x + C$

25. $\int \operatorname{csch} x\coth x\,dx = -\operatorname{csch} x + C$

26. $\int \tanh x\,dx = \ln\cosh x + C$

27. $\int \coth x\,dx = \ln|\sinh x| + C$

28. $\int \operatorname{sech} x\,dx = \tan^{-1}\sinh x + C = \sin^{-1}\tanh x + C$

29. $\int \operatorname{csch} x\,dx = \ln|\tanh(x/2)| + C$

Trigonometric Integrals

30. $\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

31. $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

32. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

33. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

34. $\int \tan^2 x\,dx = \tan x - x + C$

35. $\int \cot^2 x\,dx = -\cot x - x + C$

36. $\int \cos^3 x\,dx = -\frac{1}{3}\sin^3 x + \sin x + C$

37. $\int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x + C$

38. $\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

39. $\int \csc^3 x\,dx = -\frac{1}{2}\csc x\cot x - \frac{1}{2}\ln|\csc x + \cot x| + C$

40. $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C$

41. $\int \cot^3 x\,dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

42. $\int \sec^n ax\,\tan ax\,dx = \frac{1}{na}\sec^n ax + C;\ n \neq 0$

43. $\int \csc^n ax\cot ax\,dx = -\frac{1}{na}\csc^n ax + C;\ n \neq 0$

44. $\int \frac{dx}{1 + \sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

45. $\int \frac{dx}{1 - \sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + C$

46. $\int \frac{dx}{1 + \cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

47. $\int \frac{dx}{1 - \cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

48. $\int \sin mx\cos nx\,dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C;\ m^2 \neq n^2$

49. $\int \sin mx\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C;\ m^2 \neq n^2$

50. $\int \cos mx\cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C;\ m^2 \neq n^2$

Reduction Formulas for Trigonometric Functions

51. $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

52. $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

53. $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx;\ n \neq 1$

54. $\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx;\ n \neq 1$

55. $\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx;\ n \neq 1$

56. $\int \csc^n x\,dx = -\frac{\csc^{n-2}x\cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx;\ n \neq 1$

57. $\int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx;\ m \neq -n$

58. $\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx;\ m \neq -n$

59. $\int x^n\sin ax\,dx = -\frac{x^n\cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\,dx;\ a \neq 0$

60. $\int x^n\cos ax\,dx = \frac{x^n\sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx;\ a \neq 0$

Integrals Involving $a^2 - x^2$; $a > 0$

61. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

62. $\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + C$

63. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$

64. $\int x^2\sqrt{a^2 - x^2}\,dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a} + C$

65. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{1}{x}\sqrt{a^2 - x^2} - \sin^{-1}\frac{x}{a} + C$

66. $\int \frac{x^2}{\sqrt{a^2 - x^2}}\,dx = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

67. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x + a}{x - a}\right| + C$

Integrals Involving $x^2 - a^2$; $a > 0$

68. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln|x + \sqrt{x^2 - a^2}| + C$

69. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C$

70. $\int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$

71. $\int x^2\sqrt{x^2 - a^2}\,dx = \frac{x}{8}(2x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8}\ln|x + \sqrt{x^2 - a^2}| + C$

72. $\int \frac{\sqrt{x^2 - a^2}}{x^2}\,dx = \ln|x + \sqrt{x^2 - a^2}| - \frac{\sqrt{x^2 - a^2}}{x} + C$

73. $\int \frac{x^2}{\sqrt{x^2 - a^2}}\,dx = \frac{a^2}{2}\ln|x + \sqrt{x^2 - a^2}| + \frac{x}{2}\sqrt{x^2 - a^2} + C$

74. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C$

75. $\int \frac{dx}{x(x^2 - a^2)} = \frac{1}{2a^2}\ln\left|\frac{x^2 - a^2}{x^2}\right| + C$

Integrals Involving $a^2 + x^2$; $a > 0$

76. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln\left(x + \sqrt{a^2 + x^2}\right) + C$

77. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right) + C$

78. $\int \frac{dx}{x\sqrt{a^2 + x^2}} = \frac{1}{a}\ln\left|\frac{a - \sqrt{a^2 + x^2}}{x}\right| + C$

79. $\int \frac{dx}{x^2\sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{a^2 x} + C$

80. $\int x^2\sqrt{a^2 + x^2}\, dx = \frac{x}{8}(a^2 + 2x^2)\sqrt{a^2 + x^2} - \frac{a^4}{8}\ln\left(x + \sqrt{a^2 + x^2}\right) + C$

81. $\int \frac{\sqrt{a^2 + x^2}}{x^2}\, dx = \ln\left|x + \sqrt{a^2 + x^2}\right| - \frac{\sqrt{a^2 + x^2}}{x} + C$

82. $\int \frac{x^2}{\sqrt{a^2 + x^2}}\, dx = -\frac{a^2}{2}\ln\left(x + \sqrt{a^2 + x^2}\right) + \frac{x\sqrt{a^2 + x^2}}{2} + C$

83. $\int \frac{\sqrt{a^2 + x^2}}{x}\, dx = \sqrt{a^2 + x^2} - a\ln\left|\frac{a + \sqrt{a^2 + x^2}}{x}\right| + C$

84. $\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}} + C$

85. $\int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 + x^2}\right) + C$

Integrals Involving $ax \pm b$; $a \neq 0, b > 0$

86. $\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C;\ n \neq -1$

87. $\int \left(\sqrt{ax + b}\right)^n dx = \frac{2}{a}\frac{\left(\sqrt{ax + b}\right)^{n+2}}{n + 2} + C;\ n \neq -2$

88. $\int \frac{dx}{x\sqrt{ax - b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax - b}{b}} + C;\ b > 0$

89. $\int \frac{dx}{x\sqrt{ax + b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax + b} - \sqrt{b}}{\sqrt{ax + b} + \sqrt{b}}\right| + C;\ b > 0$

90. $\int \frac{x}{ax + b}\, dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax + b| + C$

91. $\int \frac{x^2}{ax + b}\, dx = \frac{1}{2a^3}\left((ax + b)^2 - 4b(ax + b) + 2b^2\ln|ax + b|\right) + C$

92. $\int \frac{dx}{x^2(ax + b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax + b}{x}\right| + C$

93. $\int x\sqrt{ax + b}\, dx = \frac{2}{15a^2}(3ax - 2b)(ax + b)^{3/2} + C$

94. $\int \frac{x}{\sqrt{ax + b}}\, dx = \frac{2}{3a^2}(ax - 2b)\sqrt{ax + b} + C$

95. $\int x(ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a^2}\left(\frac{ax + b}{n + 2} - \frac{b}{n + 1}\right) + C;\ n \neq -1, -2$

96. $\int \frac{dx}{x(ax + b)} = \frac{1}{b}\ln\left|\frac{x}{ax + b}\right| + C$

Integrals with Exponential and Trigonometric Functions

97. $\int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C$

98. $\int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C$

Integrals with Exponential and Logarithmic Functions

99. $\int \frac{dx}{x\ln x} = \ln|\ln x| + C$

100. $\int x^n\ln x\, dx = \frac{x^{n+1}}{n + 1}\left(\ln x - \frac{1}{n + 1}\right) + C;\ n \neq -1$

101. $\int xe^x\, dx = xe^x - e^x + C$

102. $\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx;\ a \neq 0$

103. $\int \ln^n x\, dx = x\ln^n x - n\int \ln^{n-1} x\, dx$

Miscellaneous Formulas

104. $\int x^n\cos^{-1} x\, dx = \frac{1}{n + 1}\left(x^{n+1}\cos^{-1}x + \int \frac{x^{n+1}dx}{\sqrt{1 - x^2}}\right);\ n \neq -1$

105. $\int x^n\sin^{-1} x\, dx = \frac{1}{n + 1}\left(x^{n+1}\sin^{-1}x - \int \frac{x^{n+1}dx}{\sqrt{1 - x^2}}\right);\ n \neq -1$

106. $\int x^n\tan^{-1} x\, dx = \frac{1}{n + 1}\left(x^{n+1}\tan^{-1}x - \int \frac{x^{n+1}dx}{x^2 + 1}\right);\ n \neq -1$

107. $\int \sqrt{2ax - x^2}\, dx = \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x - a}{a}\right) + C;\ a > 0$

108. $\int \frac{dx}{\sqrt{2ax - x^2}} = \sin^{-1}\left(\frac{x - a}{a}\right) + C;\ a > 0$

Useful trigonometric identities:

Product-to-sum
$2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$
$2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$
$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$
$2\cos\theta\sin\varphi = \sin(\theta+\varphi) - \sin(\theta-\varphi)$
$\tan\theta\tan\varphi = \dfrac{\cos(\theta-\varphi) - \cos(\theta+\varphi)}{\cos(\theta-\varphi) + \cos(\theta+\varphi)}$
$\prod_{k=1}^{n}\cos\theta_k = \dfrac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1 + \cdots + e_n\theta_n)$ where $S = \{1, -1\}^n$

Sum-to-product
$\sin\theta \pm \sin\varphi = 2\sin\left(\dfrac{\theta\pm\varphi}{2}\right)\cos\left(\dfrac{\theta\mp\varphi}{2}\right)$
$\cos\theta + \cos\varphi = 2\cos\left(\dfrac{\theta+\varphi}{2}\right)\cos\left(\dfrac{\theta-\varphi}{2}\right)$
$\cos\theta - \cos\varphi = -2\sin\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta-\varphi}{2}\right)$

Sine	$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
Cosine	$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
Tangent	$\tan(\alpha\pm\beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$
Cosecant	$\csc(\alpha\pm\beta) = \dfrac{\sec\alpha\sec\beta\csc\alpha\csc\beta}{\sec\alpha\csc\beta \pm \csc\alpha\sec\beta}$
Secant	$\sec(\alpha\pm\beta) = \dfrac{\sec\alpha\sec\beta\csc\alpha\csc\beta}{\csc\alpha\csc\beta \mp \sec\alpha\sec\beta}$
Cotangent	$\cot(\alpha\pm\beta) = \dfrac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$
Arcsine	$\arcsin x \pm \arcsin y = \arcsin\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$
Arccosine	$\arccos x \pm \arccos y = \arccos\left(xy \mp \sqrt{(1-x^2)(1-y^2)}\right)$
Arctangent	$\arctan x \pm \arctan y = \arctan\left(\dfrac{x\pm y}{1\mp xy}\right)$
Arccotangent	$\operatorname{arccot} x \pm \operatorname{arccot} y = \operatorname{arccot}\left(\dfrac{xy\mp 1}{y\pm x}\right)$